



**digital**

PARIS RESEARCH LABORATORY

## Numerical Valuation of High Dimensional Multivariate European Securities

---

March 1993

Jérôme Barraquand



---

**Numerical Valuation  
of High Dimensional Multivariate  
European Securities**

---

Jérôme Barraquand

---

March 1993

---

## Publication Notes

The author can be contacted at the following address:

Jérôme Barraquand  
Digital Equipment Corporation  
Paris Research Laboratory  
85, avenue Victor Hugo  
92563 Rueil-Malmaison Cedex  
France

`barraquand@prl.dec.com`

© Digital Equipment Corporation 1993

This work may not be copied or reproduced in whole or in part for any commercial purpose. Permission to copy in whole or in part without payment of fee is granted for non-profit educational and research purposes provided that all such whole or partial copies include the following: a notice that such copying is by permission of the Paris Research Laboratory of Digital Equipment Centre Technique Europe, in Rueil-Malmaison, France; an acknowledgement of the authors and individual contributors to the work; and all applicable portions of the copyright notice. Copying, reproducing, or republishing for any other purpose shall require a license with payment of fee to the Paris Research Laboratory. All rights reserved.

## Abstract

We consider the problem of pricing a contingent claim whose payoff depends on several sources of uncertainty. Using classical assumptions from the Arbitrage Pricing Theory, the theoretical price can be computed as the discounted expected value of future cash flows under the modified risk-neutral information process. Although analytical solutions have been developed in the literature for a few particular option pricing problems, computing the arbitrage prices of securities under several sources of uncertainty is still an open problem in many instances. In this paper, we present efficient numerical techniques based upon Monte Carlo simulation for pricing European contingent claims depending on an arbitrary number of risk sources. We introduce in particular the method of quadratic resampling (QR), a new powerful error reduction technique for Monte Carlo simulation. Quadratic resampling can be efficiently combined with classical variance reduction methods such as importance sampling to further improve the accuracy of the estimate. Our numerical experiments show that the method is practical for pricing claims depending on up to one hundred underlying assets. We also describe an implementation of the method on a massively parallel supercomputer, yielding two orders of magnitude of performance improvement over the same implementation on a desktop workstation.

## Résumé

Nous étudions le problème de l'évaluation d'un actif conditionnel dont la valeur dépend de plusieurs sources de risque. Avec les hypothèses classiques de la théorie de l'arbitrage, le prix théorique peut être calculé comme l'espérance mathématique actualisée des flux financiers futurs pour le processus modifié neutre vis-à-vis du risque. Bien que des solutions analytiques existent pour certains problèmes particuliers d'évaluation d'options, l'évaluation numérique du prix d'arbitrage des actifs multidimensionnels reste un problème ouvert à ce jour dans de nombreux cas. Nous proposons des techniques numériques efficaces basées sur la méthode de Monte Carlo pour calculer le prix d'actifs Européens dépendants d'un nombre arbitraire de sources de risque. Nous introduisons en particulier la méthode du rééchantillonnage quadratique (RQ), nouvelle technique puissante de réduction d'erreur pour la simulation de Monte Carlo. Le rééchantillonnage quadratique peut être combiné avec les méthodes classiques de réduction de variance telles que l'échantillonnage par importance pour réduire encore l'erreur d'estimation. Les résultats expérimentaux montrent l'efficacité de cette approche pour évaluer des actifs conditionnels dépendants d'une centaine d'actifs sous-jacents. Nous présentons aussi une implantation de la méthode sur une machine massivement parallèle, permettant de gagner deux ordres de grandeur en vitesse de calcul par rapport à une implantation sur station de travail.

## Keywords

Finance, arbitrage, pricing, options, multidimensional contingent claim, Monte Carlo method, importance sampling, quadratic resampling, massively parallel computing

## Acknowledgements

The author wishes to thank Philippe Henrotte for bringing to his attention several relevant results in the recent financial literature, and Olivier Multon for his assistance during the parallel implementation.

## Contents

1	Introduction	1
1.1	Background . . . . .	1
1.2	Multidimensional pricing models . . . . .	1
1.3	European versus American instruments . . . . .	3
2	Relation to other work	5
3	Arbitrage pricing of European securities	7
3.1	Diffusion model of information process . . . . .	7
3.2	European securities . . . . .	8
3.3	Arbitrage Pricing . . . . .	9
4	Monte Carlo valuation	10
4.1	Numerical integration on the real line . . . . .	10
4.2	Multidimensional integration . . . . .	11
4.3	Monte Carlo method and importance sampling . . . . .	13
4.4	Quasi-random sampling . . . . .	15
5	Quadratic resampling	17
6	Experimental results	20
6.1	A test case . . . . .	20
6.2	Comparison with classical Gaussian rules . . . . .	26
6.3	Massively parallel implementation . . . . .	28
7	Conclusion	30
	References	32





## 1 Introduction

### 1.1 Background

Since the seminal work of Black and Scholes [Black and Scholes, 73] and Merton [Merton, 73] in the early 1970s, the arbitrage principle underlying option valuation theory has been extended to a broad range of other financial instruments (see e.g. [Ross, 76], [Cox and Rubinstein, 85]). Indeed, any security whose returns are contractually related to the returns on some other security or group of securities can theoretically be valued using the same arbitrage principle. This is the case in particular of warrants, convertible bonds, but also common stocks, ordinary bonds, and most other types of contractual instruments. In some cases, explicit closed form analytical formulas for the computation of the arbitrage price can be derived from this theory. In particular, the original paper of Black and Scholes provides a closed form solution for a European option on a single common stock. Unfortunately, few other cases can be solved analytically, and computing the arbitrage price often requires numerical simulations. Following an idea initially presented in an early edition of [Sharpe, 85], Cox, Ross, and Rubinstein [Cox, Ross, and Rubinstein, 79] [Cox and Rubinstein, 85] developed a discrete model for the valuation of an American option on a single stock that can be easily computed numerically. However, the effective implementation of the arbitrage principle is not always such an easy task, and may sometimes become intractable. For example, when a contingent claim depends on many sources of uncertainty, there is no known general tractable algorithm for computing accurately the arbitrage price.

### 1.2 Multidimensional pricing models

There are several reasons motivating the development of efficient methods for multidimensional contingent claim pricing. We briefly review some of them below, but the list is certainly not exhaustive. The tremendous development of financial engineering during the past decade can be expected to continue, and new types of securities requiring multidimensional modeling are likely to appear at a sustained pace in the future.

- *Complex Over The Counter warrants*

Many instruments depending on several underlying securities are being proposed on Over The Counter markets. One can buy options on the best performance of several securities, or on the spread between different market indexes. In foreign exchange markets, complex instruments are being proposed that enable investors to diversify their portfolios internationally while protecting them from the relative variations of several different currencies. In commodity markets, long-dated options on portfolios of futures contracts with various maturities are being proposed to protect investors from long term evolutions of the underlying commodity and of its convenience yield.

- *Path-dependent instruments*

Specialized institutions are now proposing options that depend not only on the current value of the underlying asset, but on the whole history of underlying asset prices until expiration date. Well known examples are lookback options (e.g. options on the minimum value of the underlying for a given period of time), and Asian options (e.g. options on the average value of the underlying for a given period of time). These path-dependent instruments can be viewed as multidimensional contingent claims on the sequence of underlying asset prices until the expiration date. Other examples of path-dependent instruments are Mortgage-Backed Securities (MBS), which depend on the past history of interest rates, and certain life insurance contracts, such as Single Premium Deferred Annuities (SPDA) and Guaranteed Interest Contracts (GIC).

- *Interest rate related instruments*

A new generation of term structure models [Heath, Jarrow, and Morton, 92] has recently been developed. These models can account for non-parallel evolutions of the term structure, i.e. discrepancies in the evolutions of short and long-term rates. According to these models, the whole term structure of interest rate is a multidimensional underlying asset for interest rate contingent claims. Examples of interest rate contingent claims include all kinds of fixed-income securities such as bonds and convertible bonds, bond options, swaps, caps, floors, collars, swaptions, but also mortgage-backed securities and SPDAs [Fabozzi, 87].

- *Models with stochastic parameters*

It is well known that the volatility of the underlying asset cannot be assumed constant for long-term options. Recently, new option pricing models have been developed that attempt to take into account the stochastic nature of volatility (see e.g. [Wiggins, 87], [Dothan, 87], [Hull and White, 88]). In these models, the option price depends on two sources of uncertainty: the underlying asset itself, and its volatility. Similarly, the interest rates cannot be assumed constant for a long period of time. Therefore, pricing long-dated options requires modeling the uncertainty arising for interest rate variations. Examples of commodity or stock option pricing models accounting for uncertainties in stochastic interest rates can be found in [Merton, 73], [Jarrow, 87]. Of course, as described above, fixed income securities and related derivative instruments also require modeling of the stochastic nature of interest rates. More generally, any parameter in the stochastic model of the underlying asset that cannot be assumed predictable constitutes a new source of uncertainty that cannot be neglected for long-dated instruments.

- *Quality delivery options in futures contracts*

The seller of a futures contract must deliver at expiration date some predefined underlying asset. Sometimes, the contract specifies a series of acceptable underlying assets, in which case the seller will deliver the cheapest one. In essence, the futures contract is a multidimensional option on the minimum value of all the possible deliverable assets. Consideration for the quality option is especially important in the case of interest rate futures, where a number of bonds with different durations and characteristics can often be delivered [Cheng, 87], [Boyle, 89].

- *Pricing of insurance contracts*

The Arbitrage Pricing Theory can be applied to the pricing of insurance policies. In particular, as described above, some life insurance policies can be viewed as pure interest rate contingent claims (see e.g. [Stanton, 89]). More generally, option pricing theory can be applied to the valuation of property/liability insurance contracts, where the strike price of the option corresponds to the deductible of the insurance policy. There is a growing academic literature on these subjects, that can be traced back to the work of [Merton, 77] and [Smith, 79]. Other applications are presented in [Kraus and Ross, 82], [Doherty and Garven, 86], [Cummins, 88] [Shimko, 92].

- *Assets and Liabilities Management*

Large corporations bear many different kinds of risks in the course of their activities. Some of these risks can be hedged on financial markets. In particular, the currency exchange risks and the interest rate risks can be hedged in many cases. For financial institutions such as major banks and life insurance companies, this financial risk is the only major source of uncertainty of the business. In order to minimize this risk, the corporation must immunize itself as much as possible against variations of the economic value of the firm as a function of the sources of uncertainty. This economic value is simply the difference between the present value of assets and the present value of liabilities. Given a model of the commercial and financial policy of the firm, these present values can be computed as contingent claims on the economic variables representing the risk (e.g. interest rates and currency exchange rates).

- *Capital budgeting models in corporate investment decision making*

When assessing the opportunity to invest in a new project, a financial manager must determine the *Net Present Value* (NPV) of the project, i.e. the economic value of all the possible future cash flows generated by the project. Several economic and commercial variables are generally relevant for building a model of the probability distribution of the future cash flows. In particular, when building the business plan for introducing a new product to the market, one must estimate the possible evolutions of the parameters associated with the product, as functions of the predefined marketing policy. These parameters can be the market size, the market share of the company, the fixed and variable costs, the expected price. Also, global economic variables such as gross national product, rate of inflation, or interest rates may be relevant. According to [Brealey and Myers, 91], the net present value can be computed as the expected value of all future cash flows, discounted at the opportunity cost of capital. Computing the NPV can be viewed as pricing a multidimensional contingent claim on the relevant economic and commercial parameters. See [Mason and Merton, 85] for a review of applications of option pricing theory to corporate finance.

### 1.3 European versus American instruments

For pricing purposes, financial assets can be divided into two major classes. The first class is that of assets whose future cash-flows cannot be influenced by decisions from the holder taken

after the purchase date. Most assets traded on financial markets belong to this class. We will call European instruments all financial assets belonging to this first class. In particular, stocks, bonds, futures contracts, European options, swaps, caps, floors, mortgage-backed securities are European instruments. The second class is that of assets whose cash flows can be influenced a posteriori by the holder. American options belong to this class. As another example, the crediting policy of a Life Insurance company selling SPDAs greatly influences the present value of the liabilities of the company. This crediting policy can be adjusted by the company after signature of the contracts with the policyholders. Therefore, the liability associated with the sale of SPDA can be viewed as an American security. We will call American instruments all financial assets belonging to this second class.

Following the general theory of arbitrage pricing, the theoretical price of a European contingent claim is the discounted expected value of its future cash flows under the so-called “risk-neutral” probability distribution of the underlying economic factors [Harrison and Kreps, 79] [Harrison and Pliska, 81], [Duffie, 88], [Karatzas and Shreve, 88]. Mathematically, computing the arbitrage price reduces to computing an integral (sum) over the space of the underlying economic factors. When the dimension of the space of the underlying economic factors is small, standard techniques for numerical integration can be used. In some cases, the integral can even be computed analytically (e.g. Black-Scholes formula). However, the computational complexity of evaluating the integral is clearly exponential in the dimension of the space.

In this paper, we present efficient numerical techniques based upon Monte Carlo simulation that enable us to compute accurate approximations of realistic European pricing problems in a time quadratic in the number of underlying economic factors. Combining classical techniques of importance sampling with a new approximation method called *quadratic resampling*, we can compute the theoretical prices of the most complex instruments within tens of seconds on a PC or a workstation, and within a fraction of a second on currently available massively parallel supercomputers. The method has been tested for pricing problems with up to 100 underlying factors.

The price of an American claim is the maximum over all possible cash flow monitoring strategies of the associated present values of cash flows. For example, the price of an American option is the maximum over all possible early exercise strategies of the corresponding present values. Since the space of cash flow monitoring strategies is generally huge, direct maximization of the present value is rarely practical (see [Bossaerts, 89] for a discussion). However, when the underlying economy is modeled as a Markov process, one can use the Bellman principle of dynamic programming to compute the optimal monitoring strategy. American options are typically priced using a discrete approximation of the dynamic programming principle. This is the case in particular of the CRR model [Cox, Ross, and Rubinstein, 79] for American stock option pricing. This approach becomes however impractical when the underlying economic space has many dimensions, since the dynamic programming algorithm requires a memory space exponential in the number of dimensions. This fact is known as the “curse of dimensionality” problem [Bellman, 57] for dynamic programming. Unfortunately, the techniques developed in this paper for European claim pricing do not generalize simply to the case of American claims.

This paper is organized as follows. In Section 2, we relate our contribution to previous work in Monte Carlo integration and asset pricing. In Section 3, we recall the usual assumptions on the stochastic processes governing the evolution of securities prices, and the main result of the Arbitrage Pricing Theory. In Section 4, we briefly review Monte Carlo techniques for the numerical valuation of multidimensional integrals, and describe in particular modern importance sampling methods. In Section 5, we present the method of quadratic resampling. In Section 6, we present numerical results demonstrating the efficiency of the quadratic resampling method when combined with appropriate importance sampling schemes. We also describe a massively parallel implementation of our Monte Carlo procedure. Finally, in Section 7, we discuss the capabilities and limitations of the approach.

## 2 Relation to other work

Good reviews of the Monte Carlo method and different variance reduction techniques such as antithetic variables, covariates, stratified sampling, importance sampling can be found in many sources such as [Hammersley and Handscomb, 64], [Zaremba, 68], [Haber, 70], [Kalos and Whitlock, 86], [Decker, 91] and references thereof. [Stroud, 71] is a comprehensive source of results regarding multidimensional Gaussian quadrature rules. All our experiments on Gaussian rules are based on methods proposed in [Stroud, 71]. The additive importance sampler presented in this paper is described in [Dantzig and Glynn, 89] and [Dantzig and Infanger, 91] in the context of portfolio optimization and other operations research applications.

The application of the Monte Carlo method to option pricing was first presented in [Boyle, 77], in the context of claims contingent to a single underlying asset. It has then been used by several authors for the valuation of path dependent contingent claims. In particular, the method has been used for pricing mortgage-backed securities (see [Schwartz and Torous, 89], [Hutchinson and Zenios, 91]). Interestingly, the model described in [Schwartz and Torous, 89] could be solved faster by direct finite-difference approximation of the arbitrage partial differential equation. Indeed, although the problem is initially path-dependent as a function of interest rates, it can be made path-independent by including two additional state variables in the model. These new variables aggregate all the necessary information on past values of interest rates. The final model is four-dimensional, which is still tractable for direct numerical integration. This technique of including new state variables to remove path dependence is described in much detail in [Stanton, 89]. Many problems of path-dependence can be solved that way. The Monte Carlo method, although much slower than direct numerical integration for problems with few dimensions, is very flexible and simple to implement. This is why it has been used in many problems of path-dependence that could be solved faster by direct numerical integration. Another application of the Monte Carlo method to the valuation of Single Premium Deferred Annuities contracts is described in [Stanton, 89].

The valuation of options depending of several underlying assets has been studied by several authors. [Brennan and Schwartz, 79] addresses the problem of pricing options under two sources of risk by direct finite-difference approximation of the generalized Black-Scholes equation. In this example the two sources of risk are the short term and the long term

interest rates. The approach is clearly limited to a few assets, since the memory space requirements and the computation time are both exponential in the number of underlying assets. [Boyle, Evnine, and Gibbs 89] developed a multinomial lattice method for pricing multidimensional options, in the spirit of the approach outlined in [Cox and Rubinstein, 85]. According to the authors, the computation becomes very burdensome for more than two assets. In fact, the multinomial lattice approach can be viewed as a finite-difference approximation of the generalized Black-Scholes equation using an explicit Euler scheme and an appropriate change of variables [Brennan and Schwartz, 78].

[Stulz, 82] presents an analytical solution to the problem of pricing a European option on the maximum or minimum of two underlying assets. The analytical solution is generalized in [Johnson, 87] to the case of an arbitrary number of assets, taking as given the cumulative multivariate normal distribution function. [Boyle, 89] and [Boyle and Tse, 90] developed an approximate method for the same problem. Although the problem is solved analytically in [Johnson, 87], the approximate method does not require preliminary computation of the cumulative multivariate normal distribution function. To the best of our knowledge, the above problem of option pricing on the maximum or minimum of several assets is the only case reported in the literature where solutions to high-dimensional (say, more than 5) problems have been developed.

Our contribution to the problem of multidimensional contingent claim pricing reported in this paper is twofold:

–First, we show that a proper implementation of the Monte Carlo method makes the problem of European contingent claims pricing tractable, even when the number of underlying assets is very large (up to 100 in our experiments).

–Second, we present a set of three techniques for improving the accuracy of the Monte Carlo estimation.

- We introduce an original error reduction technique, quadratic resampling, that considerably improves the accuracy of Monte Carlo integration. Quadratic resampling can be used for any multidimensional integration problem, and is computationally practical for integrals with up to a few hundred dimensions. It can be combined with classical variance reduction techniques such as importance sampling to further improve the accuracy of the computation. It is shown that, given a random vector having arbitrarily many coordinates and an arbitrary joint distribution, the expected value of any polynomial of degree two or less in the vector coordinates is computed exactly when using quadratic resampling.
- We introduce a deterministic Quasi-Monte Carlo method which does not require the use of pseudo-random numbers. For integrals in spaces with few dimensions, this Quasi-Monte Carlo sampling method is considerably more accurate than pseudo-random sampling. This method combines the advantages of direct numerical integration in low-dimensional spaces with those of Monte Carlo integration in high-dimensional spaces.

- We show that with currently available massively parallel technology, multidimensional contingent claim prices can be computed two orders of magnitude faster than on a desktop workstation. Therefore, since the accuracy of the Monte Carlo estimate is proportional to the square root of the number of samples, the massively parallel implementation is one order of magnitude more accurate than the workstation implementation for a given computing time.

### 3 Arbitrage pricing of European securities

The arbitrage pricing theory is described in many textbooks. We refer the reader to [Duffie, 92] for a comprehensive presentation.

#### 3.1 Diffusion model of information process

We model the economy as a finite-dimensional vector of real-valued state variables  $X(t) = (x_1(t), \dots, x_n(t))$ , called *factors*, representing all the information available to investors at time  $t$ . By definition,  $X(t)$  is known at time  $t$ . However,  $X(t)$  does not usually give enough information to investors to allow perfect forecasting of the future evolutions of security markets. Therefore,  $X(\tau)$ ,  $\tau > t$  is a stochastic process.

By construction,  $X(t)$  is a Markov process, i.e. future values of vector  $X$  are independent of past values of vector  $X$ , conditionally to the knowledge of  $X$  today. Indeed, if future values of  $X$  were to depend not only on  $X(t)$  but also on past values, this would mean that  $X(t)$  does not represent all relevant information at time  $t$ , contradicting its definition.

Since  $X(t)$  represents all information available to agents at time  $t$ , in a frictionless market, prices of securities must be deterministic functions of time and  $X(t)$ . It is said that securities are contingent claims on the state variable  $X(t)$ .

For the sake of simplicity, we will assume that the information process  $X(t)$  is a *diffusion process*. However, our results on Monte Carlo valuation described in the next sections apply to more general types of stochastic processes. If  $X(t)$  is a diffusion process, it is a solution of a stochastic differential equation of the type:

$$dX = A(X(t), t)dt + B(X(t), t)dW \quad (1)$$

or equivalently:

$$\forall i \in [1, n], \quad dx_i = a_i(x_1, \dots, x_n, t)dt + \sum_{j=1}^n b_{ij}(x_1, \dots, x_n, t)dw_j$$

The vector  $A$  is called the drift of process  $X$ .  $A$  is the derivative of the expected value of  $X$ . The matrix  $\Gamma = BB^T$  is the derivative of the covariance of  $X$ .  $W = (w_1, \dots, w_n)$  is a  $n$ -dimensional standard Brownian motion.

Often, the variables  $x_i$  are prices of securities available on the market, and are therefore strictly positive processes. The expected increments and covariance of increments are then expressed in relative values:

$$\forall i \in [1, n], \quad \frac{dx_i}{x_i} = \mu_{x_i}(x_1, \dots, x_n, t)dt + \sum_{j=1}^n v_{ij}(x_1, \dots, x_n, t)dw_j$$

with

$$\mu_{x_i} = a_i/x_i, \quad v_{ij} = b_{ij}/x_i$$

The matrix  $V = (v_{ij})_{(i,j) \in [1,n]^2}$  is called volatility matrix, and the covariance of relative returns is the matrix  $\Sigma = (\sigma_{ij})_{(i,j) \in [1,n]^2}$ :

$$\Sigma = VV^T$$

We have the following relationship:

$$\gamma_{ij} = \sigma_{ij}x_i x_j$$

### 3.2 European securities

A security is called *European security* iff future cash flows cannot be influenced by decisions from the holder taken after the purchase date (besides of course selling back the security). Then the cash flows are only functions of time and information.

The cash flow generated by a European security  $\mathcal{C}$  during the time interval  $dt$ , assuming  $\mathcal{C}$  is held indefinitely, will be denoted by  $f_{\mathcal{C}}(X(t), t)dt$ .

If the price  $C(t)$  of a European security  $\mathcal{C}$  is positive, we can define the instantaneous relative cash flow rate or *dividend yield* as

$$d_{\mathcal{C}}(X(t), t) = f_{\mathcal{C}}(X(t), t)/C(t)$$

The aggregate cash flow  $F_{\mathcal{C}}(t_0, t)$  from the purchase date  $t_0$  to any future date  $t$  follows the equation:

$$F_{\mathcal{C}}(t_0, t_0) = 0, \quad dF_{\mathcal{C}} = f_{\mathcal{C}}dt = d_{\mathcal{C}}Cdt$$

The total *gain process*  $G_{\mathcal{C}}$ , sum of the security's liquidative value and the aggregate cash flow is:

$$G_{\mathcal{C}}(t_0, t) = C(t) + F_{\mathcal{C}}(t_0, t)$$

Let  $\mu_{\mathcal{C}}$  denote the expected capital rate of return of  $\mathcal{C}$ :

$$\mu_{\mathcal{C}}dt = E\left(\frac{dC}{C}\right)$$

The expected total rate of return of  $\mathcal{C}$ , i.e. the expected rate of return of the total gain process is:

$$\mu_{G_{\mathcal{C}}} = \mu_{\mathcal{C}} + d_{\mathcal{C}}$$



The dividend yield of the money market account  $\mathcal{L}$ , called short term interest rate, is denoted by  $r(X)$ . If the proceeds of the money market account are continuously reinvested, the total gain process  $L(t_0, t)$  follows the equation:

$$L(t_0, t_0) = 1, \quad dL = r(X)L(t_0, t)dt$$

or equivalently:

$$L(t_0, t) = \exp \left( \int_{t_0}^t r(X(\tau))d\tau \right)$$

### 3.3 Arbitrage Pricing

For any risk factor  $x_i$ , let  $e(C/x_i)$  denote the *elasticity* of price  $C$  to  $x_i$ :

$$e(C/x_i) = \frac{x_i}{C} \frac{\partial C}{\partial x_i}$$

The major result of the Arbitrage Pricing Theory is the following. There exist numbers  $\lambda_1, \dots, \lambda_n$ , called market prices for risk, such that for any security  $\mathcal{C}$ , the following relationship holds:

$$\mu_{G_C} = r + \sum_{i=1}^n e(C/x_i) \lambda_i$$

Furthermore, if factor  $x_i$  is traded, the market price of risk for factor  $x_i$  is the expected total rate of return on  $x_i$  in excess of the riskless interest rate. In particular, if all  $x_i$  are traded, we have:

$$\mu_{G_C} - r = \sum_{i=1}^n e(C/x_i) (\mu_{G_{x_i}} - r) = \sum_{i=1}^n e(C/x_i) (\mu_{x_i} + d_{x_i} - r)$$

Using the properties of diffusion processes, the above results lead to the following partial differential equation, called Black-Scholes equation:

$$-\frac{\partial C}{\partial t} = (d_C - r)C + \sum_{i=1}^n \frac{\partial C}{\partial x_i} (\mu_{x_i} - \lambda_i) x_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 C}{\partial x_i \partial x_j} \gamma_{ij} \quad (2)$$

If factor  $x_i$  is the price of a traded security, the term  $\mu_{x_i} - \lambda_i$  is simply  $r - d_{x_i}$ . In particular, if all factors are traded, Black-Scholes equation simplifies to:

$$-\frac{\partial C}{\partial t} = (d_C - r)C + \sum_{i=1}^n \frac{\partial C}{\partial x_i} (r - d_{x_i}) x_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 C}{\partial x_i \partial x_j} \gamma_{ij} \quad (3)$$

We see that the above equation does not depend on the market prices for risk. Therefore, we can replace the information process  $X$  by the so-called *risk-neutral* information process  $\tilde{X}$  for which all market prices for risk are zero.

Using a theorem known under the name of Feynman-Kac Formula, we can represent explicitly the solution of the above equation:

$$C(X(t), t) = \tilde{E}_t \left( \int_t^\infty \frac{f_C(X(\tau), \tau)}{L(t, \tau)} d\tau \right)$$

where  $\tilde{E}_t$  represents the expectation under the fictitious risk-neutral information process  $\tilde{X}$ .

Under suitable technical conditions, and using the linearity of the expectation operator:

$$C(X(t), t) = \int_t^\infty \tilde{E}_t \left( \frac{f_C(X(\tau), \tau)}{L(t, \tau)} \right) d\tau$$

In particular, if a security only yields cash flows  $f_C(X(t_1), t_1), \dots, f_C(X(t_k), t_k)$  at discrete dates  $t_1 < \dots < t_k$ , we have for any time  $t < t_1$ :

$$C(X(t), t) = \sum_{i=1}^k \tilde{E}_t \left( \frac{f_C(X(t_i), t_i)}{L(t, t_i)} \right)$$

Therefore, the problem of computing the price of a security with  $k$  discrete cash-flows  $f_C(X(t_1), t_1), \dots, f_C(X(t_k), t_k)$  reduces to that of computing the prices of each of the  $k$  securities  $C_{t_1}, \dots, C_{t_k}$  with respective single cash flows  $f_C(X(t_1), t_1), \dots, f_C(X(t_k), t_k)$ .

In practice, securities only yield cash flows at discrete times. In the following, we will consider without loss of generality a security  $C$  with a single cash flow  $f_C(X(T))$  at a given *expiration date*  $T$ . Then,

$$C(X(t), t) = \tilde{E}_t \left( \frac{f_C(X(T))}{L(t, T)} \right)$$

Let  $p(X)$  be the risk-neutral probability density of  $X(T)$  knowing  $X(t)$ . We have:

$$C(X(t), t) = \int_{R^n} \left( \frac{f_C(X)}{L(t, T)} \right) p(X) dX \quad (4)$$

The practical problem of pricing a European security under  $n$  sources of uncertainty reduces to that of computing the above  $n$ -dimensional integral.

## 4 Monte Carlo valuation

### 4.1 Numerical integration on the real line

We consider the problem of evaluating the simple integral:

$$I(f) = \int_a^b f(x) dx$$

on the compact interval  $[a, b]$  for any function  $f$ . When the above integral cannot be computed analytically, the natural solution consists of approximating the integral by a weighted sum of values of  $f$  over a finite number of samples  $x^1, \dots, x^m$ . The simplest quadrature rule is the trapezoidal rule:

$$I(f) \approx \frac{b-a}{2m}f(a) + \frac{b-a}{m} \sum_{i=1}^{m-1} f(x^i) + \frac{b-a}{2m}f(b)$$

with

$$\forall i \in [1, m-1], \quad x^i = a + \frac{i}{m}(b-a)$$

In the above formula,  $m$  is an integer chosen large enough for accurate approximation of the integral.

More complex formulas such as Simpson rule or Romberg integration have a higher order of accuracy for smooth integrands (see [Stoer and Burlisch, 80] for a review).

A technique of particular interest is that of *Gaussian quadrature*. If the function  $f$  is the product of a polynomial  $g$  of degree  $2m-1$  and some known *kernel*  $W$ , then we can find points  $x^1, \dots, x^m$  and some weights  $w^1, \dots, w^m$  independent of  $g$  such that the integral:

$$I_W(g) = \int_a^b f(x)dx = \int_a^b g(x)W(x)dx$$

is *exactly* equal to the sum

$$\sum_{i=1}^m g(x^i)w^i$$

## 4.2 Multidimensional integration

The basic idea underlying numerical quadrature generalizes simply to  $n$ -dimensional integration problems. We consider an  $n$ -dimensional hypercube  $H = [a^1, b^1] \times \dots \times [a^n, b^n]$  and the quadrature problem:

$$I(f) = \int_H f(x_1, \dots, x_n) dx_1 \dots dx_n \quad (5)$$

For each coordinate  $x_i$ , we choose  $m_i$  samples  $x_i^1, \dots, x_i^{m_i}$ . This generates a lattice of  $n$ -dimensional samples containing the  $m_1 \times \dots \times m_n$  points

$$\forall (j_1, \dots, j_n) \in [0, m_1] \times \dots \times [0, m_n], \quad X^{j_1 \dots j_n} = (x_1^{j_1}, \dots, x_n^{j_n})$$

We can find appropriate weights  $w^{j_1, \dots, j_n}$  such that the sum:

$$\sum_{(j_1, \dots, j_n) \in [0, m_1] \times \dots \times [0, m_n]} f(x_1^{j_1}, \dots, x_n^{j_n}) w^{j_1, \dots, j_n}$$

approximates the integral  $I$ . For example, if we have for the  $i^{th}$  dimension a one-dimensional integration formula of the form:

$$\sum_{j=1}^{m_i} f(x_i^j) w_i^j$$

we can build the Cartesian product rule with weights

$$w^{j_1, \dots, j_n} = \prod_{i=1}^n w_i^{j_i}$$

In particular, if  $m_1 = \dots = m_n = m$ , the number of samples is  $m^n$ . Therefore, evaluating the sum is a problem requiring a computation time exponential in the number of dimensions  $n$ . Hence, direct numerical integration is impractical for high-dimensional problems.

Practical computation of a high-dimensional integral requires to choose fewer samples  $X^1, \dots, X^M$  and weights  $w^1, \dots, w^M$ , such that the total number of samples  $M$  is polynomial in the dimension  $n$ . If the samples and weights are chosen carefully, the sum

$$\sum_{i=1}^M f(X^i) w^i$$

approximates the integral  $I$ .

Choosing appropriate samples and weights is in general a difficult (i.e. intractable) problem. However, in many practical situations, the functions  $f$  to be integrated have specific properties that make the problem tractable. In particular, security pricing problems of the type (4) lend themselves surprisingly well to numerical integration.

The main idea described above underlying Gaussian quadrature can be extended to multi-dimensional spaces. While classical Cartesian product rules require a number of samples exponential in the dimension, there exist non-Cartesian rules requiring only a polynomial number of samples. Let us consider a kernel  $W(x_1, \dots, x_n)$  defined over  $R^n$ . In general, it can be shown (see [Stroud, 71]) that there exist a Gaussian rule exact for all functions  $f$  that can be represented as the product of  $W$  with a polynomial of degree  $m$ , and that requires at most

$$M = \frac{(m+n)!}{m!n!}$$

samples, number which is indeed polynomial in  $n$  for a given  $m$ . For example, there exist a formula exact for polynomials of degree two in  $R^n$  requiring only  $M = (n+1)(n+2)/2$  samples. In fact, for second degree polynomials, there even exist a formula with only  $n+1$  samples in dimension  $n$ . In practice however, those rules have some negative coefficients  $w^i$ , and are thus very unstable numerically. Most quadrature rules with only positive coefficients require at least  $2^n$  samples, and thus cannot be used for very high dimensional spaces. [Stroud, 71] presents a comprehensive review of results on generalized Gaussian quadrature rules.

We experimented with the best rules advocated by [Stroud, 71] on typical asset pricing problems in dimensions ranging from 1 to 10. In particular, we used the Cartesian product Gauss rule, exact for any polynomial of degree 3 multiplied by a Gaussian kernel  $W(X) = \exp(-\frac{1}{2}||X||^2)$ . This rule requires  $3^n$  samples. In dimension 10, this represents almost 60,000 samples. Although the rule is exact for any polynomial of degree 3, our experiments have shown (see

Section 6) that it performs very poorly of typical asset pricing problems. We also used the spherical Lobatto rule, exact for any polynomial of degree 5, and requiring  $2^{n+1} - 1 = 2047$  samples in dimension 10. The results, presented in Section 6, show that these Gaussian quadrature rules are inadequate for asset pricing problems in more than a few dimensions.

A alternative way of choosing the samples is to generate them randomly, using a pseudo-random number generator. The resulting integration technique, known under the name of *Monte Carlo* integration, is quite widespread and simple to implement. We present in the next subsection the general principle underlying Monte Carlo integration.

### 4.3 Monte Carlo method and importance sampling

We consider the hypercube  $H = [0, 1]^n$  without loss of generality, and a uniformly distributed random vector  $X$  over  $H$ . By definition, the expected value of the variable  $f(X)$  is the integral  $I(f)$  defined by equation (5):

$$E^u(f(X)) = I(f) = \int_H f(x_1, \dots, x_n) dx_1 \dots dx_n$$

where  $E^u$  denotes the expectation under the uniform distribution. The principle of Monte Carlo integration is to generate  $M$  independent random samples  $X^1, \dots, X^M$  of the vector  $X$  and to compute the empirical mean:

$$\frac{1}{M} \sum_{i=1}^M f(X^i)$$

By the law of large numbers, this sample mean is approximately equal to the integral  $I(f)$  for  $M$  large enough.

This above crude Monte Carlo method converges very slowly even in dimension 1, and usually does not converge in a reasonable amount of time in high-dimensional spaces.

One technique for accelerating convergence consists of performing an appropriate change of variables in the integral. The technique is known in the literature as *importance sampling*. Let us assume that we know a *positive* function  $\tilde{f}$  over  $H$ , such that  $\tilde{f}$  approximates  $f$  in some sense, and that the integral:

$$I(\tilde{f}) = \int_H \tilde{f}(x_1, \dots, x_n) dx_1 \dots dx_n$$

is easy to compute. Then, we can consider the probability distribution whose density is

$$q = \frac{\tilde{f}}{I(\tilde{f})}$$

Instead of generating uniformly distributed samples, we generate the samples  $X^1, \dots, X^M$  with the importance sampling density  $q$ . To do so, we assume that we can find a change of variable in  $R^n$   $Y = (y_1, \dots, y_n)$  such that:

$$|\det(dY/dX)| = q$$

Therefore,

$$I(f) = \int_H f dX = \int_H \frac{f}{q} q dX = \int_{Y(H)} \frac{f}{q} dY = E^q\left(\frac{f}{q}\right)$$

where  $E^q$  denotes the expectation for the distribution of density  $q$ . Equivalently:

$$I(f) = I(\tilde{f}) E^q\left(\frac{f}{\tilde{f}}\right)$$

In particular, when  $f$  is positive, if we choose  $\tilde{f} = f$ ,  $f/\tilde{f}$  is always equal to 1, hence  $E^q(f/\tilde{f}) = 1$  and there is no need to generate any samples. However, the normalizing factor  $I(\tilde{f}) = I(f)$  is precisely what we aimed at estimating in the first place. In practice,  $\tilde{f}$  must be chosen in such a way that it is approximately proportional to  $f$ , while still significantly easier to integrate than  $f$ .

There is no easy systematic way of choosing the appropriate importance sampling density  $q$ . However, the integration problem is often naturally expressed in such a way that  $f$  is proportional to a special probability density  $p$ :

$$f = gp$$

For problem (5),  $p$  is the function constant over the hypercube  $H$ , density of the uniform distribution. In general, when the problem at hand is to compute the expected value of a random variable  $g(X)$  where  $X$  has density  $p$ , we have  $f = gp$ .  $p$  is then called the *natural importance sampling* density, and the method of importance sampling simply consists in *simulating*  $X$  with its natural probability distribution  $p$ .

For asset pricing problems of the form (4), the risk-neutral probability density  $p(X)$  of the information process  $X$  is often a good candidate for importance sampling. We generate samples  $X^1, \dots, X^M$  with the risk neutral probability distribution, and approximate the asset price by the average discounted cash flow:

$$C(X(t), t) \approx \frac{1}{M} \sum_{i=1}^M \frac{fc(X^i)}{L(t, T)}$$

This technique is called risk-neutral importance sampling.

Recently, [Dantzig and Glynn, 89] and [Dantzig and Infanger, 91] have proposed the method of additive importance sampling, an interesting paradigm for choosing importance sampling functions. They consider a natural probability density function  $p(X)$  in  $R^n$ , which makes the  $n$  coordinates independent two by two:

$$p(X) = \prod_{i=1}^n p_i(x_i)$$

The corresponding distribution could be for example the uniform distribution over an hypercube  $H$  ( $p_i = 1/(b_i - a_i)$ ) or the multivariate standard normal distribution ( $p_i = \exp(-x_i^2/2)/\sqrt{2\pi}$ ).

We consider the problem of computing the integral:

$$I_p(f) = \int_{R^n} f(X)p(X)dX$$

The method of additive importance sampling consists of identifying  $n$  positive functions  $f_1(x_1), \dots, f_n(x_n)$  such that their sum

$$\tilde{f}(x_1, \dots, x_n) = \sum_{i=1}^n f_i(x_i)$$

approximate the function  $f$ . The integral of  $\tilde{f}$  is easier to compute than that of  $f$ , since it is a sum of monodimensional integrals:

$$I_p(\tilde{f}) = \sum_{i=1}^n \int_{-\infty}^{+\infty} f_i(x)p_i(x)dx$$

Then

$$I_p(f) = \int_{R^n} \frac{f}{\tilde{f}} \tilde{f} p dX = \sum_{i=1}^n \int_{R^n} \frac{f(X)}{\tilde{f}(X)} f_i(x_i) p(X) dX$$

The problem reduces to that of computing each of the  $n$  integrals:

$$\forall i \in [1, n], \quad I^i = \int_{R^n} \frac{f(X)}{\tilde{f}(X)} f_i(x_i) p(X) dX$$

Due to the particular structure of the above integral, it is easy to generate samples with a density proportional to  $f_i(x_i)p(x)$ . Then, if  $\tilde{f}$  is actually a good approximation of  $f$ , the additive importance sampler will converge substantially faster than the simple natural importance sampler. The importance sampling density is

$$q = p\tilde{f}/I_p(\tilde{f})$$

It is the product of an additive function and of a multiplicative function of the variables  $x_1, \dots, x_n$ .

#### 4.4 Quasi-random sampling

Monte Carlo methods combined with appropriate importance sampling schemes are experimentally efficient. However, generating the samples randomly is not necessarily the best way of obtaining an accurate quadrature rule. Indeed, the efficiency of Monte Carlo methods does not depend on the random nature of the samples, but on their equidistribution properties. Deterministic techniques for generating samples often perform better than random techniques. For historical reasons, deterministic techniques for choosing the samples are called *Quasi Monte Carlo* techniques. There is a fairly large literature on Quasi-Monte Carlo methods, that we will not present here. We refer the reader to [Zaremba, 68] and [Haber, 70] for a review.

We developed a very simple method for generating deterministically equidistributed samples, that happens to perform very well in practice. For low dimensional problems (say, problems in dimension  $n < 5$ ), this approach is considerably more efficient than classical methods based upon random number generation. On higher dimensional problems, the method yields results similar to that of classical Monte Carlo methods. Hence, we can get the best of both worlds: reasonable efficiency for high-dimensional problems, and high efficiency for low dimensional problems.

We consider again problem (5). We assume that an appropriate importance sampling density  $q$  has been chosen. After the corresponding change of variable  $Y(X)$  such that  $|\det(dY/dX)| = q$ , we are faced with the following problem:

$$I(g) = \int_{[0,1]^n} g dY$$

with  $g = f/q$ . Instead of generating randomly uniformly distributed samples, we choose to sample the  $n$ -dimensional hypercube deterministically. We first choose an appropriate quantization step  $h$  for each coordinate, say  $h = 1/2^k$  for a positive integer  $k$ . Since we make our calculations on a 32 bits computer, a simple choice is  $k = 32$ . A quantized point  $Y$  in the hypercube is then represented by a sequence of  $n$   $k$ -bits integers  $(y_1, \dots, y_n)$ . We consider the  $kn$ -bits integer:

$$L(y_1, \dots, y_n) = \sum_{i=1}^n y_i 2^{k(i-1)}$$

For each number  $y_i$ , we can write the representation of  $y_i$  in base 2:

$$y_i = \sum_{j=1}^k y_i^j 2^{j-1}$$

where  $y_i^j \in \{0, 1\}$ . Then:

$$L(y_1, \dots, y_n) = \sum_{i=1}^n \sum_{j=1}^k y_i^j 2^{k(i-1)+j-1}$$

The mapping  $L$  is a one-to-one mapping from the quantized hypercube  $[0, 2^k]^n$  onto the integer interval  $[0, 2^{nk}]$ . Intuitively, this mapping corresponds to scanning the quantized hypercube in *lexicographic* order.

Alternatively, we can consider the mapping in *Peano* order, defined as:

$$P(y_1, \dots, y_n) = \sum_{j=1}^k \sum_{i=1}^n y_i^j 2^{n(j-1)+i-1}$$

If we consider the variables

$$\forall j \in [1, k], \quad z_j = \sum_{i=1}^n y_i^j 2^{i-1}$$



spanning the hypercube  $[0, 2^n[$ , we get:

$$P(y_1, \dots, y_n) = \sum_{j=1}^k z_j 2^{n(j-1)}$$

Hence, the Peano mapping can be viewed geometrically as a one-to-one mapping between the hypercube  $[0, 2^k[$  and the hypercube  $[0, 2^n[$ . In our experiments, the lexicographic mapping  $L$  and the Peano mapping  $P$  gave similar results. For  $n = 1$ , both the lexicographic and the Peano scan reduce to the identity function over the interval  $[0, 2^k[$ .

Given any of those two mappings, say  $L$ , we can now construct a sampling of the hypercube by first sampling regularly the interval  $[0, 2^{nk}[$ , and then taking the inverse of the mapping.

Given the desired number of samples  $M$ , we perform the integer division:

$$2^{nk} = MA + B$$

with  $B < M$ . Then, we sample regularly the interval  $[0, 2^{nk}[$  with the  $M$  numbers

$$\forall r \in [1, M], \quad Y_r = rA$$

The corresponding samples in the hypercube  $[0, 2^k[$  are the  $M$  points  $(y_1^r, \dots, y_n^r)$  verifying

$$L(y_1^r, \dots, y_n^r) = Y_r = rA$$

If  $M$  is properly chosen, the samples are correctly distributed over the hypercube. In practice, we choose  $M$  of the form  $M = p^\alpha$ , where  $p$  is a prime integer different from 2, say  $p = 3$ . More advanced number theoretic methods of this type are described in [Stroud, 71].

In dimension  $n = 1$ , the sampling is simply a regular sampling of the interval  $[0, 1]$ . Hence, the quadrature formula is almost identical to a regular trapezoidal formula, much more accurate than a formula generated by a random sampling. More generally, this quasi Monte Carlo method has the advantage of not introducing spurious randomness in low dimensional integration problems. For high-dimensional problems, experiments have shown that it performs similarly to classical Monte Carlo methods.

## 5 Quadratic resampling

We consider again the problem of computing the integral:

$$I_p(f) = \int_{R^n} f(X) p(X) dX$$

where  $p$  is a natural probability density function, and  $f$  any integrable function. In particular, the general quadrature problem (5) is a special case of the above problem,  $p$  being the uniform density over the hypercube  $H$ . If we consider the random vector  $X$  with density  $p$ , this is equivalent to the problem of computing the expected value

$$I_p(f) = E(f(X))$$

We assume that the  $n$  first order moments have been precomputed (either analytically or through Monte Carlo simulation, using one of the importance sampling techniques described above).

$$\forall i \in [1, n], \quad I_p(x_i) = E(x_i) = \int_{R^n} x_i p(X) dX$$

The vector whose coordinates are  $E(x_i)$  is denoted  $E(X)$ , and called expected value of  $X$ . We also assume that the  $n^2$  second order moments have been precomputed.

$$\forall (i, j) \in [1, n]^2, \quad I_p(x_i x_j) = E(x_i x_j) = \int_{R^n} x_i x_j p(X) dX$$

The matrix of second order moments is denoted  $E(XX^T)$ . By definition, the covariance matrix of  $X$  is:

$$K_X = E((X - E(X))(X - E(X))^T) = E(XX^T) - E(X)E(X)^T$$

$K_X$  is a non-negative symmetric matrix. Without loss of generality, we will assume that  $K$  is positive. Indeed, if  $K_X$  is degenerate, we can always diagonalize it in an orthonormal basis, perform the corresponding change of variables in the integral  $I_p$ , and discard all degenerate dimensions. Then, the above quadrature problem is transformed in a lower dimensional non-degenerate quadrature problem.

Since  $K_X$  is non-negative symmetric, we can find a square root, i.e. a matrix denoted  $\sqrt{K_X}$  such that:

$$K_X = \sqrt{K_X} \sqrt{K_X}^T$$

This can be done either through  $LU$  decomposition, or through diagonalization of  $K_X$  in a orthonormal basis. Since  $K_X$  is positive,  $\sqrt{K_X}$  is regular (i.e. has an inverse).

We then consider a sequences of samples  $X^1, \dots, X^M$ , generated either deterministically or randomly. For example, they can be generated randomly so as to follow the distribution of density  $p$ . In general, we can choose any appropriate importance sampling density  $q$  for generating the samples. Then, we can define the empirical mean approximating the integral  $I_p(f)$  as:

$$\overline{f(X)} = \sum_{i=1}^M f(X^i) w^i$$

with

$$\forall i \in [1, n], \quad w^i = \frac{1}{\sum_{j=1}^M p(X^j)/q(X^j)} \frac{p(X^i)}{q(X^i)}$$

The normalizing factor for  $w^i$ 's has been chosen so as to have:

$$\sum_{i=1}^M w^i = 1$$

In particular, the empirical mean is:

$$\bar{X} = \sum_{i=1}^M X^i w^i$$

We can define similarly the empirical covariance:

$$\overline{K_X} = \overline{(X - \bar{X})(X - \bar{X})^T} = \overline{XX^T} - \bar{X}\bar{X}^T$$

More explicitly:

$$\overline{K_X} = \sum_{i=1}^M (X^i - \bar{X})(X^i - \bar{X})^T w^i = \sum_{i=1}^M X^i X^{iT} w^i - \bar{X}\bar{X}^T$$

If the samples are generated randomly, the law of large numbers applies. Therefore, for a sufficiently large number  $M$  of samples, the empirical mean  $\bar{X}$  and the empirical covariance  $\overline{K_X}$  will be arbitrarily close to the real mean  $E(X)$  and the real covariance  $K_X$ . Therefore, since the set of regular matrices is open, the empirical covariance  $\overline{K_X}$  must be non-singular for  $M$  sufficiently large. If the samples are not generated randomly, there exist deterministic sampling techniques guaranteeing the same convergence property of the empirical mean and covariance to the real mean and covariance, and the same result applies.

Since  $\overline{K_X}$  is non-singular, its square root  $\sqrt{\overline{K_X}}$  is also non-singular.

We define the gain matrix

$$H = \sqrt{K_X} \sqrt{\overline{K_X}}^{-1}$$

and the new random variable:

$$Y = H(X - \bar{X}) + E(X)$$

We consider the  $M$  samples  $Y^i = H(X^i - \bar{X}) + E(X)$ . For this particular sampling, we get:

$$\bar{Y} = \sum_{i=1}^M Y^i w^i = E(X)$$

Similarly:

$$\overline{K_Y} = \overline{(Y - \bar{Y})(Y - \bar{Y})^T} = \overline{(H(X - \bar{X}))(H(X - \bar{X}))^T} = H\overline{K_X}H^T$$

Using the expressions above for  $H$  and  $\overline{K_X}$  as products of square root matrices we get:

$$\overline{K_Y} = K_X$$

Hence, the empirical first and second order moments using the samples  $Y^i$  are exactly equal to the real first and second order moments of  $X$ . In particular, the empirical mean of any polynomial  $f$  of degree two or less in the variables  $x_1, \dots, x_n$  verifies:

$$\overline{f(Y)} = \sum_{i=1}^M f(Y^i) w^i = E(f(X))$$

We can state:

*Any numerical quadrature formula generated through quadratic resampling is exact for any polynomial of degree two or less*

In the above sense, the method of quadratic resampling can be viewed as a particular Gaussian integration rule of degree 2, following the terminology of [Stroud, 71].

The method of quadratic resampling can be summarized as follows:

- 1) Precompute the expected value  $E(X)$  and covariance matrix  $K_X$  of the random vector  $X$  with density  $p$ , if they are not available in analytical form.
- 2) Choose an importance sampling density  $q(x)$  well-suited to the problem at hand, the simplest choice being  $q = p$ .
- 3) Sample the random vector  $X$  using any available technique, i.e. Monte Carlo sampling or deterministic sampling, to obtain a sequence  $X^1, \dots, X^M$ . Define the empirical quadrature formula:

$$\overline{f(X)} = \sum_{i=1}^M f(X^i) w^i$$

Compute the empirical mean  $\bar{X}$  and empirical covariance matrix  $\bar{K}_X$ . Choose  $M$  large enough so that  $\bar{K}_X$  is non-singular.

- 4) Compute the matrix  $H = \sqrt{\bar{K}_X} \sqrt{\bar{K}_X}^{-1}$
- 5) Compute the modified samples  $Y^i = H(X^i - \bar{X}) + E(X)$ , to obtain the modified empirical quadrature formula:

$$\overline{f(Y)} = \sum_{i=1}^M f(Y^i) w^i$$

## 6 Experimental results

### 6.1 A test case

We implemented the different numerical quadrature schemes presented above on a test case described below.

We consider two real numbers  $\sigma$  and  $\rho$  with  $-1/(n-1) \leq \rho \leq 1$ , and the non-negative symmetric matrix  $K = (k_{ij})_{(i,j) \in [1,n]^2}$ , such that

$$\forall i \in [1, n], \quad k_{ii} = \sigma^2$$

and

$$\forall (i, j) \in [1, n]^2, i \neq j, \quad k_{ij} = \rho \sigma^2$$

We consider a random vector  $X = (x_1, \dots, x_n)^T \in R^n$ , following a multivariate normal distribution with zero mean and covariance matrix  $K$ .

Let us assume that  $K$  is of rank  $k \leq n$ , and let us compute a square root  $V$  of  $K$ , i.e. a  $n \times k$  matrix such that:

$$K = VV^T$$

We consider the random vector  $Z$ :

$$Z = (V^T V)^{-1} V^T X$$

By construction,  $X = VZ$ , and  $Z$  follows a  $k$ -dimensional standard normal distribution.

We can always take  $k = n$  by completing the matrix  $V$  with columns of zeros, and consider the quadrature problem:

$$I(f) = E(f(X)) = \int_{R^n} f(X) p_X(X) dX = \int_{R^n} f(VZ) p_Z(Z) dZ$$

where  $p_X$  and  $p_Z$  are the densities of  $X$  and  $Z$ . Hence:

$$I(f) = \int_{R^n} f(VZ) \frac{\exp(-||Z||^2/2)}{\sqrt{2\pi^n}} dZ$$

We choose the particular expression for  $f$ :

$$f(x_1, \dots, x_n) = \max_{i=1}^n \exp(x_i)$$

With this choice, we obtain a three parameters quadrature problem:

$$I(n, \sigma, \rho) = \int_{R^n} \max_{i=1}^n \exp\left(\sum_{j=1}^n v_{ij} z_j\right) \frac{\exp(-\sum_{k=1}^n z_k^2/2)}{\sqrt{2\pi^n}} dz_1 \dots dz_n$$

The above problem admits the following financial interpretation. We consider a stock market including  $n$  dividend-free stocks with prices  $S_1, \dots, S_n$  following a jointly lognormal diffusion process with any arbitrary expected yearly returns  $\mu_i$ , and a covariance matrix of yearly returns equal to  $K$ . We have:

$$\forall i \in [1, n], \quad \frac{dS_i}{S_i} = \mu_i dt + \sum_{j=1}^n v_{ij} dw_j$$

where  $W = (w_1, \dots, w_n)^T$  is an  $n$ -dimensional standard Brownian motion. It can be shown that the solution of the above equation is

$$\forall i \in [1, n], \quad S_i(t) = S_i(0) \exp\left((\mu_i - k_{ii}/2)t + \sum_{j=1}^n v_{ij} w_j(t)\right)$$

Using  $k_{ii} = \sigma^2$  and setting  $x_i(t) = \sum_{j=1}^n v_{ij} w_j(t)$ , we get:

$$S_i(t) = S_i(0) \exp\left((\mu_i - \sigma^2/2)t\right) \exp(x_i(t))$$

We assume that the annualized interest rate  $r$  is constant and known at time 0. Therefore, the risk-neutral information process is obtained by replacing all  $\mu_i$ 's in the above equations by  $r$ . If we consider the price  $C$  at time zero of a contingent security with a single cash flow  $g(S_1, \dots, S_n)$  at date  $T$ , the Feynman-Kac formula yields:

$$C = \int_{R^n} \exp(-rT) g(S_1, \dots, S_n) \frac{\exp(-\sum_{k=1}^n z_k^2/2)}{\sqrt{2\pi}^n} dz_1 \dots dz_n \quad (6)$$

with

$$S_i = S_i(0) \exp((r - \sigma^2/2)T) \exp((\sum_{j=1}^n v_{ij} z_j) \sqrt{T})$$

If the contingent claim is an option with zero strike entitling its holder to choose any one of the  $n$  securities  $S_1, \dots, S_n$  at expiration date, we have:

$$g(S_1, \dots, S_n) = \max_{i=1}^n S_i$$

Indeed, any clever investor will choose the security with the highest market value at time  $T$ .

Finally, if we assume that the initial prices of the stocks at time zero are all equal to 1, and that the cash flow is exactly one year hence ( $T = 1.0$ ), we get:

$$S_i = \exp(r - \sigma^2/2) \exp(x_i)$$

Hence:

$$C = \exp(-\sigma^2/2) I(n, \sigma, \rho)$$

The above quadrature problem  $I(n, \sigma, \rho)$  can be interpreted as an option pricing problem. Although this problem can be solved analytically using the  $n$ -dimensional multivariate normal distribution function (see [Johnson, 87]), it is an appropriate testbed for numerical integration methods.

We tested four different Monte Carlo procedures, to be described below, on the above test case for different values of the parameters. All the four methods use importance sampling. Indeed, for large values of  $n$ , a crude Monte Carlo method using uniformly distributed samples  $z_1, \dots, z_n$  would take an astronomical number of samples to converge. The simplest importance sampler is the Natural Importance Sampler (NIS), which consists of generating  $z_1, \dots, z_n$  following a  $n$ -dimensional standard normal distribution. Since the normal distribution has the property of central symmetry at the origin, we also use in the four methods the technique of Antithetic Variables (AV). Instead of generating randomly  $M$  samples, we generate  $M/2$  samples  $Z^1, \dots, Z^{M/2}$  and take for the remaining samples  $Z^{M/2+1}, \dots, Z^M$  the opposite values  $-Z^1, \dots, -Z^{M/2}$  ( $M$  is assumed even for simplicity). The technique of antithetic variables applied to normal samples guarantees in particular that the empirical first order moments (i.e. the expected value of  $Z$ ) are exactly equal to their theoretical values (0 here). If the function  $f$  has some amount of correlation with the coordinates  $z_i$ , the technique will reduce the variance of the samples, and thus improve the accuracy of the Monte Carlo estimate. This is the case in particular for the function  $f$  chosen in our test case. In general, it is the case for most asset

pricing problems, since the cash-flows generated by securities are often monotonous functions of the underlying economic factors.

We now describe the four methods:

**Method 1: AV-NIS** Method 1 is simply the one just described. It uses the technique of antithetic variables, and uses the natural importance sampler.

**Method 2: AV-NIS-AIS** Method 2 uses in addition to Method 1 the Additive Importance Sampler (AIS) described in the previous sections. The functions  $f_i$  chosen are:

$$\forall i \in [1, n], \quad f_i(z_i) = \exp(v_{ii}z_i)$$

**Method 3: AV-NIS-QR** Method 3 uses in addition to Method 1 the technique of Quadratic Resampling (QR). Since the moments of the multivariate normal distribution are known analytically, we do not need to precompute  $E(Z)$  and  $K_Z$ :

$$E(Z) = 0, \quad K_Z = I_n$$

where  $I_n$  is the identity matrix in  $R^n$ .

**Method 4: AV-NIS-AIS-QR** Method 4 uses in addition to Method 2 the technique of Quadratic Resampling.

These four methods were tested on the quadrature problem  $I(n, \sigma, \rho)$  for different values of the parameters. All tests used a sample size  $M = 4000$ . Table 1 shows the standard errors for the four methods in dimension  $n = 10$  using  $\sigma = 1$  and four different values of the correlation parameter  $\rho = -1/9, 0, 1/2, 1$ . The value  $\sigma = 1$  corresponds to a volatility of 100 percent per year for each of the 10 prices  $S_1, \dots, S_{10}$ .

Table 2 shows the standards errors, again in dimension  $n = 10$ , using  $\sigma = 0.1$ . This represents a volatility of 10 percent per year for each of the underlying assets prices.

The results presented in Tables 1 and 2 illustrate the relative efficiency of the different methods on 10-dimensional problems. For problems with high volatility ( $\sigma = 100\%$ ), quadratic resampling (Method 3, AV-NIS-QR) reduces the error by a factor ranging from 2 (low correlation) to 3 (high correlation) (as compared to Method 1, AV-NIS). The method of additive importance sampling also reduces the error by a factor of 2 (low correlation) to 1.2 (high correlation) (Method 2 / Method 1). Interestingly, the error reduction power of quadratic resampling and additive importance sampling can be *combined* (in Method 4), yielding an error reduction factor of approximately 4 over Method 1. This corresponds in terms of computing time to a speedup of 16 for a given accuracy. The quadratic resampling method works best on problems exhibiting a high correlation between variables  $\rho \geq 0.5$ , while the additive importance sampling method works best on problems with low correlation between variables.

$\rho$	$I(10, 1.0, \rho)$	Method 1	Method 2	Method 3	Method 4
-1/9	5.91	0.0734	0.0421	0.0423	0.0260
0.0	5.62	0.0697	0.0311	0.0424	0.0204
0.5	4.18	0.0534	0.0219	0.0287	0.0125
1.0	1.65	0.0265	0.0213	0.0083	0.0054

Table 1: Standard errors in dimension  $n = 10$  for  $\sigma = 100\%$ 

$\rho$	$I(10, 0.1, \rho)$	Method 1	Method 2	Method 3	Method 4
-1/9	1.178	0.00109	0.00104	0.00050	0.00051
0.0	1.168	0.00102	0.00097	0.00047	0.00047
0.5	1.119	0.00070	0.00065	0.00032	0.00032
1.0	1.005013	0.00016	0.00013	0.00000048	0.00000032

Table 2: Standard errors in dimension  $n = 10$  for  $\sigma = 10\%$ 

For lower volatilities ( $\sigma = 10\%$ ), the additive importance sampler performs poorly. The method of quadratic resampling yields an error reduction factor ranging from 2 for problems with low correlation to several hundreds for problems with high correlation. Indeed, the cash-flow function is closer to its second order Taylor expansion for lower volatilities. Since quadratic resampling yields exact results for functions that are equal to their second order Taylor expansion, it is likely to give more accurate results for problems with low volatilities.

All calculations were performed on a DECstation 5000 model 200 desktop workstation, using a MIPS R3000 microprocessor at a clock rate of 25 Mhz. All calculations were performed in double precision floating point arithmetics. The simulation program was written in the C programming language, and compiled without any optimization. The resulting computing times for each run of 4000 samples were just below 1 second.

Tables 3 and 4 present the same results in dimension  $n = 5$ . The conclusions are similar. For high volatilities, quadratic resampling reduces the error by a factor ranging from 2 (low correlation) to 5 (high correlation). When combined with additive importance sampling, the total error reduction factor ranges from 5 to 10, corresponding to a reduction factor in computing time ranging from 25 to 100. For lower volatilities, the additive importance sampler performs again poorly. The method of quadratic resampling yields an error reduction factor ranging from 5 for problems with low correlation to several hundreds for problems with high correlation. In Table 4, the error reduction of Method 4 over Method 1 ranges from 5 to 1000, corresponding to a reduction factor in computing time ranging from 25 to 1 million. The resulting computing times for each run of 4000 samples were about 1/3 of a second.

Finally, Tables 5 and 6 present results in dimension  $n = 100$ . For high volatilities (Table 5), quadratic resampling reduces the error by a factor ranging from 1.25 (low correlation) to 3.5 (high correlation). When combined with additive importance sampling, the total error



$\rho$	$I(5, 1.0, \rho)$	Method 1	Method 2	Method 3	Method 4
-1/4	4.39	0.0719	0.0365	0.0230	0.0141
0.0	4.07	0.0594	0.0186	0.0255	0.0088
0.5	3.29	0.0448	0.0102	0.0205	0.0058
1.0	1.65	0.0284	0.0183	0.0067	0.0028

Table 3: Standard errors for  $n = 5$  and  $\sigma = 100\%$ 

$\rho$	$I(5, 0.1, \rho)$	Method 1	Method 2	Method 3	Method 4
-1/4	1.141	0.00133	0.00126	0.00027	0.00027
0.0	1.126	0.00117	0.00110	0.00024	0.00024
0.5	1.090	0.00079	0.00073	0.00017	0.00016
1.0	1.005013	0.00017	0.00011	0.00000040	0.00000017

Table 4: Standard errors for  $n = 5$  and  $\sigma = 10\%$ 

$\rho$	$I(100, 1.0, \rho)$	Method 1	Method 2	Method 3	Method 4
0.0	13.58	0.1235	0.0984	0.0994	0.0818
0.5	7.94	0.0850	0.0694	0.0486	0.0422
1.0	1.65	0.0301	0.0294	0.0087	0.0084

Table 5: Standard errors for  $n = 100$  and  $\sigma = 100\%$ 

$\rho$	$I(100, 0.1, \rho)$	Method 1	Method 2	Method 3	Method 4
0.0	1.286	0.000901	0.000887	0.000698	0.000696
0.5	1.1975	0.000588	0.000579	0.000457	0.000456
1.0	1.005013	0.000169	0.000165	0.00000051	0.00000049

Table 6: Standard errors for  $n = 100$  and  $\sigma = 10\%$

reduction factor ranges from 1.5 to 3.6, corresponding to a reduction factor in computing time ranging from 2.3 to 13. For lower volatilities, the additive importance sampler performs again poorly. The method of quadratic resampling yields an error reduction factor ranging from 1.3 for problems with low correlation to several hundreds for problems with high correlation. In Table 6, the error reduction of Method 4 over Method 1 ranges from 1.3 to 350, corresponding to a reduction factor in computing time ranging from 1.7 to 100,000.

The resulting computing times for each run of 4000 samples were about 1 minute and 10 seconds. In general, the computation time is quadratic in the dimension  $n$ , since the computation of each sample requires to compute the transformation  $X = VZ$ , and  $V$  is a  $n \times n$  matrix.

To illustrate typical computing time in practical cases, let us consider the computation of  $I(10, 1.0, 0.0)$  using Method 4. The result, presented in Table 1, is 5.62, and has a standard error of approximately 0.02. Using the central limit theorem, we see that the accuracy for a confidence level of 99.995% is approximately 0.08 (4 times the standard error). In order to make the last digit significant, an accuracy of 0.01 (8 times higher) is required. Hence, the computation time for 0.01 accuracy would be  $8 \times 8 = 64$  times higher. The result would be obtained in about 1 minute of computation on the same hardware platform, with 256000 samples. With Method 1, the same result would be obtained in about 12 minutes of computation, using more than 2.5 millions of samples. Not surprisingly, Table 2 shows that problems with lower volatility require much less computing time. Even for low correlations, three digit accuracy is achieved in a small fraction of a second, and four digit accuracy in a couple of seconds (the error margin of Method 3 for  $\rho = 0.0$  is about  $4 \times 0.00047 \approx 0.002$  for a confidence level of 99.995% using only 4000 samples, i.e. a computation time of 1 second).

In general, the computing time may vary from several minutes to a fraction of a second, depending of the characteristics of the covariance matrix, and on the accuracy required. Since the accuracy of Monte Carlo methods is proportional to the square root of the number of samples, getting one additional significant digit requires a computing time 100 times longer on a given hardware platform.

## 6.2 Comparison with classical Gaussian rules

We implemented several classical Gaussian integration rules proposed in [Stroud, 71] and experimented with them on the above test case.

We present below some results for 2 different rules, designed for approximating integrals of the form:

$$I(f) = \int_{R^n} f(z_1, \dots, z_n) \frac{\exp(-\sum_{k=1}^n z_k^2/2)}{\sqrt{2\pi}^n} dz_1 \dots dz_n$$

**Gauss1** is a classical Cartesian product rule of the form described in Section 4.2. It is exact if  $f$  is a polynomial of degree 3.

In dimension 1, the three sample points are  $-\sqrt{3}$ , 0, and  $\sqrt{3}$ . The corresponding weights

$\rho$	$I(10, 1.0, \rho)$	Gauss1	Gauss2	Stdev Method 4
-1/9	5.91	5.53	5.46	0.034
0.0	5.62	4.901	5.35	0.028
0.5	4.18	4.11	3.64	0.018
1.0	1.65	1.64	1.64	0.0078

Table 7: Values computed from Gaussian quadrature rules ( $n = 10$ ,  $\sigma = 100\%$ )

$\rho$	$I(10, 0.1, \rho)$	Gauss1	Gauss2	Stdev Method 4
-1/9	1.178	1.180	1.165	0.00065
0.0	1.168	1.159	1.142	0.00061
0.5	1.119	1.120	1.096	0.00041
1.0	1.005013	1.005013	1.005013	0.0000004

Table 8: Values computed from Gaussian quadrature rules ( $n = 10$ ,  $\sigma = 10\%$ )

are  $1/6$ ,  $2/3$ , and  $1/6$ . In dimension  $n$ , the total number of samples is therefore  $3^n$ .

**Gauss2** is a so-called spherical Lobatto rule (see [Stroud, 71]). It is exact for polynomials of degree 5. The total number of samples in dimension  $n$  is  $2^{n+1} - 1$

In dimension 10, the number of samples required for the Cartesian product rule Gauss1 is 59049. The number of samples required for the spherical Lobatto rule Gauss2 is 2047. Results are presented in Tables 7 and 8. In order to compare these Gaussian rules with the Monte Carlo method, the last column of the tables shows the standard error of Method 4 above, for the exact same number of samples as the Lobatto rule, i.e. 2047 in this case. The results from Table 7 show that both Gaussian rules are very inaccurate for pricing problems with high volatilities and low correlations. The error is in most cases over 5%, and exceeds 10% on several examples, while the standard error of Method 4 remains under 0.5% for 2047 samples. Correct results are obtained only on problems with very high correlation. Results in Table 8 indicate that Gaussian rules perform better on problems with low volatilities, but are still not nearly as good as Monte Carlo methods for a given number of samples. The error of the Lobatto rule ranges from 1% to over 2%, while the error of Method 4 remains below 0.05% with the exact same number of samples. The Cartesian product Gauss rule Gauss1 is about as accurate as Method 4 for low volatilities, but it requires 30 times more samples.

Tables 9 and 10 present results in dimension  $n = 5$ . Gauss1 requires 243 samples, while Gauss2 requires 63 samples.

Results are presented in Tables 9 and 10. As before, the last column of the tables shows the standard error of Method 4 above, for the exact same number of samples as the Lobatto rule, i.e. 63 in this case. The error reported in Table 9 (high volatilities) ranges from 5%

$\rho$	$I(5, 1.0, \rho)$	Gauss1	Gauss2	Stdev Method 4
-1/4	4.39	4.30	4.35	0.116
0.0	4.07	3.78	3.87	0.077
0.5	3.29	3.30	2.93	0.048
1.0	1.65	1.64	1.64	0.020

Table 9: Values computed from Gaussian quadrature rules ( $n = 5$ ,  $\sigma = 100\%$ )

$\rho$	$I(5, 0.1, \rho)$	Gauss1	Gauss2	Stdev Method 4
-1/4	1.141	1.115	1.111	0.0025
0.0	1.126	1.113	1.108	0.0022
0.5	1.090	1.086	1.073	0.0015
1.0	1.005013	1.005013	1.005013	0.0000001

Table 10: Values computed from Gaussian quadrature rules ( $n = 5$ ,  $\sigma = 10\%$ )

to 10% on most examples, while the standard error of Method 4 remains under 2% for 63 samples. Results in Table 10 indicate that Gaussian rules perform better on problems with low volatilities, but are still not nearly as good as Monte Carlo methods for a given number of samples. The error of the Lobatto rule ranges from 1% to 2%, while the error of Method 4 remains below 0.22% with the exact same number of samples.

Of course, these Gaussian rules cannot be implemented in dimension  $n = 100$ , since they would require an astronomical number (over  $10^{30}$ ) of samples.

In conclusion, we see that classical Gaussian quadrature rules do not stand the comparison with the best Monte Carlo methods, even on relatively low-dimensional problems (e.g.  $n = 5$ ). Gaussian integration methods are only useful on problems with very few dimensions (say 1 to 3).

### 6.3 Massively parallel implementation

We implemented the Monte Carlo procedure on a DECmpp 12000 massively parallel deskside workstation with 4K (4096) processors. We present below comparative computing times with a DECstation 5000 model 200 workstation.

We consider again  $n$  stocks with a covariance matrix of relative returns of the form described in the previous subsection. The price of any claim contingent to the  $n$  stocks with a single cash-flow date is computed from formula (6).

We considered several examples for the cash-flow function  $g$ . The first example was that of an option on the best performance of the  $n$  stocks, with a non-zero strike price  $H$ . Then, the

function  $g$  is:

$$g(S_1, \dots, S_n) = \max \left( 0, \max_{i=1}^n S_i - H \right) \quad (7)$$

The second example was that of an option on a market index (CAC 40) representing the weighted average of  $n = 40$  stocks:

$$g(S_1, \dots, S_{40}) = \max \left( 0, \sum_{i=1}^{40} \alpha_i S_i - H \right)$$

Several other forms of cash-flow functions were also tested such as the quality delivery option of futures contracts:

$$g(S_1, \dots, S_n) = \min_{i=1}^n S_i$$

We implemented for the above pricing problem the Method 1, AV-NIS, on the DECmpp. The program was written in the MASPAR MPL language, a parallel extension of C. The parallelization paradigm for Monte Carlo quadrature algorithms is particularly simple on such SIMD (Single Instruction Multiple Data) computers. It consists of distributing on different processors the computation of different samples, and finally adding the results obtained by all processors. We give below experimental computing times for the first example of cash flow function. The parallelization speedups obtained for all other examples were strictly similar. In this first example using function (7), the initial prices of the  $n = 10$  stocks were all set to  $S_i(0) = 40.0$  dollars, the strike price was  $H = \$40.0$ , the interest rate was set to  $r = 7\%$ , the time to expiration  $T = 200$  days. The volatility  $\sigma$  was set to 30%, and the correlation  $\rho$  to 50%. In order to reach a 1/2 cent accuracy at a confidence level of 99.995%, we used 16 millions of samples. The computed price was  $C = \$12.26$ .

The computation time on the 4K DECmpp was 62 seconds, using single precision floating point arithmetics. In comparison, the computation time for the same problem on a DECstation 5000-200 in single precision and using all available compiler optimizations was 31 minutes. This represents a speedup factor of 30. In double precision, the speedup was 23. We estimate that the speedup on a full-size (16K) DECmpp 12000 would be exactly four times the speedup obtained on the 4K DECmpp, since the Monte Carlo method can be fully parallelized. Then, the speedup factor would be about 90 in double precision and 120 in single precision. These figures are clearly not specific to this particular example, and apply in fact to any Monte Carlo integration problem where the number of samples required is higher than the number of available processors.

We can conclude that the massively parallel implementation yields a speedup factor of roughly two orders of magnitude for most Monte Carlo integration problems.

Method 3, AV-NIS-QR, parallelizes exactly in the same fashion after precomputation of the modified samples. For the particular example above, the error reduction factor using quadratic resampling was 2.3, yielding a speedup factor of more than 5 over Method 1, AV-NIS.

We also compared Method 1 and Method 3 on the same example in dimension  $n = 100$ , all other parameters being unchanged. The call price is \$20.23. The speedup factor obtained

through quadratic resampling is only 1.5 in that case. Using Method 3 with 4000 samples, the standard error is 0.053, and the computation time is about 1 minute on a DECstation 5000. The same computation on a 4K DECmpp 12000 would be done in about 2 seconds.

Finally, we compared Method 1 and Method 3 on the same example in dimension  $n = 5$ . The call price is  $C = 9.70$ . The speedup factor obtained through quadratic resampling is 12. Using Method 3 with 4000 samples, the standard error is 0.024. The computation time is about 1/3 of a second on the DECstation, and could be done in 1/100 of a second on the DECmpp.

## 7 Conclusion

In this article, we described a systematic numerical technique for pricing arbitrarily complex European securities. Besides its obvious applications to trading and hedging in organized capital markets, European security pricing has many important applications in various areas of risk management such as assets and liabilities management and corporate investment decision making. Using this technique, we were able to compute the prices of the most complex European instruments in a few tens of seconds on a workstation, and within a fraction of a second on a massively parallel computer. We have shown that this technique outperforms previous general purpose European security pricing methods, either in the range of applicability, or in the accuracy obtained for a given computation time.

Our approach essentially relies on the well known Monte Carlo method, combined with appropriate importance sampling schemes and a new error reduction technique, that we call *quadratic resampling*. We have shown that the multidimensional quadrature problem is solved exactly for any polynomial of degree two or less when using quadratic resampling. When tested on typical multidimensional asset pricing problems, quadratic resampling yields significant error reduction. The error reduction factors ranged from 1.2 to 1000 in our experiments. The typical error reduction for problems in up to 10 dimensions was about 3, corresponding to a speedup factor of roughly one order of magnitude. The method is readily applicable to any European asset pricing problem involving an arbitrary number of variables. We have successfully implemented the method for problems with up to 100 dimensions. Besides its applications to security pricing, the method could be applied to any problem where the computation of high-dimensional integrals is required. Examples of such problems can be found for example in failure rate analysis of mechanical and electronic systems, and in the simulation of complex physical phenomena such as those occurring in nuclear physics, mechanics, and fluid dynamics.

We feel that the method presented in this paper and the experimental results obtained with it make it possible to realistically envision the use of multidimensional stochastic models for practical real-world quantitative risk management problems. This capability of computing the joint influences of several tens of risk factors such as interest rates of various terms in different currencies, equity of various kinds, and any other relevant economic variables, may dramatically increase the competitive advantage of quantitative methods over more traditional analysis techniques. An area of particular interest is that of estimating precisely the economic

value (net present value) of firms such as banks and insurance companies from a quantitative analysis of their balance sheet. However, this numerical pricing method does not generalize simply to American asset pricing problems, which cannot be reduced to multidimensional quadrature problems. Extensions of this approach for American asset pricing are left for future research.

## References

- [Bellman, 57] Bellman, R., *Dynamic Programming* Princeton University Press, 1962.
- [Black and Scholes, 73] Black, Fischer, and Myron Scholes, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 May-June 1973, 637-659.
- [Bossaerts, 89] Bossaerts, Peter, "Simulation Estimators of Optimal Early Exercise", Unpublished manuscript, Graduate School of Industrial Administration, Carnegie Mellon University, January 1989.
- [Boyle, 77] Boyle, Phelim P., "Options: A Monte Carlo Approach", *Journal of Financial Economics*, 4, 1977, 323-338.
- [Boyle, 89] Boyle, Phelim P., "The Quality Option and the timing Option in Futures Contracts", *Journal of Finance*, 44, March 1989, 101-113.
- [Boyle and Tse, 90] Boyle, Phelim P., and Y.K. Tse "An Algorithm for Computing Values of Options on the Maximum or Minimum of Several Assets", *Journal of Financial and Quantitative Analysis*, 25-2, June 1990, 215-227.
- [Boyle, Evnine, and Gibbs 89] Boyle, Phelim P., J. Evnine, and S. Gibbs "Numerical Evaluation of Multivariate Contingent Claims", *The review of Financial Studies*, 2, 1989, 241-250.
- [Brealey and Myers, 91] Brealey, Richard A., and Stewart C. Myers, *Principles of Corporate Finance*, Fourth edition, McGraw-Hill, 1991.
- [Brennan and Schwartz, 78] Brennan, M., and E. Schwartz, "Finite-Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis", *Journal of Financial and Quantitative Analysis*, 13, September 1978, 461-474.
- [Brennan and Schwartz, 79] Brennan, M., and E. Schwartz, "A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance*, 3, 1979, 133-155.
- [Cheng, 87] Cheng, S. "Pricing Models for Multiple Currency Option Bonds", Unpublished manuscript, Graduate School of Business, Stanford University.
- [Cox, Ross, and Rubinstein, 79] Cox, John C., Stephen A. Ross, and Mark Rubinstein, "Option Pricing: A Simplified Approach", *Journal of Financial Economics*, 7, September 1979, 229-263.
- [Cox and Rubinstein, 85] Cox, John C. and Mark Rubinstein, *Options Markets*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1985.
- [Cummins, 88] Cummins, J. D. "Risk-Based Premiums for Insurance Guaranty Funds", *Journal of Finance*, 43, September 1988, 823-840.



- [Dantzig and Glynn, 89] Dantzig, George, and Peter Glynn, "Parallel Processors for Planning under Uncertainty", Technical report 88-8R, Systems Optimization Laboratory, Dept. of Operations Research, Stanford University, August 1989.
- [Dantzig and Infanger, 91] Dantzig, George, and Gerd Infanger, "Multi-stage Stochastic Linear Programs for Portfolio Optimization", Technical report 91-11, Systems Optimization Laboratory, Dept. of Operations Research, Stanford University, September 1991.
- [Decker, 91] Decker, Karsten M., "The Monte Carlo Method in Science and Engineering: Theory and Application", *Computer Methods in Applied Mechanics and Engineering*, North-Holland, 1991, 463-483.
- [Doherty and Garven, 86] Doherty, N. A., and J. R. Garven, "Price Regulation in Property-Liability Insurance: A Contingent Claim Approach", *Journal of Finance*, 41, December 1986, 1031-1050.
- [Dothan, 87] Dothan, Michael U., "A Random Volatility Correction for the Black-Scholes Option-Pricing Formula", *Advances in Futures and Options Research*, Vol. 2, JAI Press Inc., 1987, 97-115.
- [Duffie, 88] Duffie, Darrell, *Security Markets: Stochastic Models*, Academic Press, Boston, 1988.
- [Duffie, 92] Duffie, Darrell, *Dynamic Asset Pricing Theory*, Princeton University Press, 1992.
- [Fabozzi, 87] Fabozzi, Frank J. and Irving M. Pollack, eds, *The Handbook of Fixed Income Securities*, second edition, Dow Jones-Irwin, Homewood, Illinois, 1987.
- [Haber, 70] Haber, S., "Numerical Evaluation of Multiple Integrals", *SIAM review*, v. 12, 1970, pp 481-526.
- [Hammersley and Handscomb, 64] Hammersley, J.M., and D.C. Handscomb, *Monte Carlo Methods*, Methuen, London, 1964.
- [Harrison and Kreps, 79] Harrison, J. Michael, and David M. Kreps, "Martingales and Arbitrage in Multiperiod Securities Markets", *Journal of Economic Theory*, Vol. 20, July 1979, 381-408.
- [Harrison and Pliska, 81] Harrison, J. Michael, and Stanley R. Pliska, "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and their Applications*, Vol. 11, 1981, 261-271. July 1979, 381-408.
- [Heath, Jarrow, and Morton, 92] Heath, David, Robert Jarrow, and Andrew Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", *Econometrica*, forthcoming, 1992.
- [Hull and White, 88] Hull, John, and Alan White, "An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility", *Advances in Futures and Options Research*, Vol. 3, JAI Press Inc., 1988, 29-61.

- [Hutchinson and Zenios, 91] Hutchinson, James M., and Stravos A. Zenios, "Financial Simulations on a Massively Parallel Connection Machine", *The International Journal of Supercomputer Applications*, Vol. 5-2, 1991, 27-45.
- [Jarrow, 87] Jarrow, Robert, A., "The Pricing of Commodity Options with Stochastic Interest Rates", *Advances in Futures and Options Research*, Vol. 2, JAI Press Inc., 1987, 19-45.
- [Johnson, 87] Johnson, Herb, "Options on the Maximum or the Minimum of Several Assets", *Journal of Financial and Quantitative Analysis*, vo. 22, 3, September 1987, 277-282.
- [Kalos and Whitlock, 86] Kalos, Malvin H., and Paula A. Whitlock, *Monte Carlo Methods*, Volume I: Basics, John Wiley and Sons, 1986.
- [Karatzas and Shreve, 88] Karatzas, Ioannis, and Steven E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, 1988.
- [Kraus and Ross, 82] Kraus, A., and S. Ross. "The Determination of Fair Profits for the Property-Liability Insurance Firm", *Journal of Finance*, 37, Sept 1982, 1015-1028.
- [Mason and Merton, 85] Mason, S. P., and R.C. Merton, "The Role of Contingent Claims Analysis in Corporate Finance", in E.I. Altman and M.G. Subrahmanyam (eds.), *Recent Advances in Corporate Finance*, Richard D. Irwin, Inc., Homewood, Ill., 1985.
- [Merton, 73] Merton, Robert, C. "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, 4, Spring 1973, 141-183.
- [Merton, 77] Merton, Robert, C. "An Analytic Derivation of the Cost of Deposit Insurance and Loan Guarantees: An Application of Modern Option Pricing Theory", *Journal of Banking and Finance*, 1, June 1977, 3-11.
- [Ross, 76] Ross, S. A., "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, 13, December 1976, 341-360.
- [Schwartz and Torous, 89] Schwartz, Eduardo S., and Walter N. Torous, "Prepayment and the Valuation of Mortgage-Backed Securities", *The Journal of Finance*, vol 44, June 1989.
- [Sharpe, 85] Sharpe, William F., *Investments*, Third edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1985.
- [Shimko, 92] Shimko, D. "The Valuation of Multiple Claim Insurance Contracts", *Journal of Financial and Quantitative Analysis*, Vol 27, 2, June 1992.
- [Smith, 79] Smith, C. "Applications of Option Pricing Analysis", In *Handbook of Financial Economics*, J.L. Bicksler, ed. Amsterdam: North Holland Publ. Co., 1979, Chapter 4.
- [Stanton, 89] Stanton, Richard, "Path Dependent Payoffs and Contingent Claim Valuation: Single Premium Deferred Annuities", Unpublished manuscript, Graduate School of Business, Stanford University, September 1989.

- [Stoer and Burlisch, 80] Stoer, J., and R. Burlisch, *Introduction to Numerical Analysis*, Springer-Verlag, NY, 1980.
- [Stroud, 71] Stroud, A. H., *Approximate Calculation of Multiple Integrals*, Prentice-Hall, 1971.
- [Stulz, 82] Stulz, Rene M., "Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications", *Journal of Financial Economics*, 10, July 1982, 161-185.
- [Wiggins, 87] Wiggins, James B., "Option Values under Stochastic Volatility: Theory and Empirical Estimates", *Journal of Financial Economics*, 19, 1987, 351-372.
- [Zaremba, 68] Zaremba, S. K., "The Mathematical Basis of Monte Carlo and Quasi-Monte Carlo Methods", *SIAM review*, v. 10, 1968, pp 303-314.



## PRL Research Reports

The following documents may be ordered by regular mail from:

Librarian – Research Reports  
Digital Equipment Corporation  
Paris Research Laboratory  
85, avenue Victor Hugo  
92563 Rueil-Malmaison Cedex  
France.

It is also possible to obtain them by electronic mail. For more information, send a message whose subject line is `help to doc-server@prl.dec.com` or, from within Digital, to `decprl : : doc-server`.

Research Report 1: *Incremental Computation of Planar Maps*. Michel Gangnet, Jean-Claude Hervé, Thierry Pudet, and Jean-Manuel Van Thong. May 1989.

Research Report 2: *BigNum: A Portable and Efficient Package for Arbitrary-Precision Arithmetic*. Bernard Serpette, Jean Vuillemin, and Jean-Claude Hervé. May 1989.

Research Report 3: *Introduction to Programmable Active Memories*. Patrice Bertin, Didier Roncin, and Jean Vuillemin. June 1989.

Research Report 4: *Compiling Pattern Matching by Term Decomposition*. Laurence Puel and Ascánder Suárez. January 1990.

Research Report 5: *The WAM: A (Real) Tutorial*. Hassan Aït-Kaci. January 1990.

Research Report 6: *Binary Periodic Synchronizing Sequences*. Marcin Skubiszewski. May 1991.

Research Report 7: *The Siphon: Managing Distant Replicated Repositories*. Francis J. Prusker and Edward P. Wobber. May 1991.

Research Report 8: *Constructive Logics. Part I: A Tutorial on Proof Systems and Typed  $\lambda$ -Calculi*. Jean Gallier. May 1991.

Research Report 9: *Constructive Logics. Part II: Linear Logic and Proof Nets*. Jean Gallier. May 1991.

Research Report 10: *Pattern Matching in Order-Sorted Languages*. Delia Kesner. May 1991.

Research Report 11: *Towards a Meaning of LIFE*. Hassan Aït-Kaci and Andreas Podelski. June 1991 (Revised, October 1992).

Research Report 12: *Residuation and Guarded Rules for Constraint Logic Programming*. Gert Smolka. June 1991.

Research Report 13: *Functions as Passive Constraints in LIFE*. Hassan Aït-Kaci and Andreas Podelski. June 1991 (Revised, November 1992).

Research Report 14: *Automatic Motion Planning for Complex Articulated Bodies*. Jérôme Barraquand. June 1991.

Research Report 15: *A Hardware Implementation of Pure Esterel*. Gérard Berry. July 1991.

Research Report 16: *Contribution à la Résolution Numérique des Équations de Laplace et de la Chaleur*. Jean Vuillemin. February 1992.

Research Report 17: *Inferring Graphical Constraints with Rockit*. Solange Karsenty, James A. Landay, and Chris Weikart. March 1992.

Research Report 18: *Abstract Interpretation by Dynamic Partitioning*. François Bourdoncle. March 1992.

Research Report 19: *Measuring System Performance with Reprogrammable Hardware*. Mark Shand. August 1992.

Research Report 20: *A Feature Constraint System for Logic Programming with Entailment*. Hassan Aït-Kaci, Andreas Podelski, and Gert Smolka. November 1992.

Research Report 21: *The Genericity Theorem and the notion of Parametricity in the Polymorphic  $\lambda$ -calculus*. Giuseppe Longo, Kathleen Milsted, and Sergei Soloviev. December 1992.

Research Report 22: *Sémantiques des langages impératifs d'ordre supérieur et interprétation abstraite*. François Bourdoncle. January 1993.

Research Report 23: *Dessin à main levée et courbes de Bézier : comparaison des algorithmes de subdivision, modélisation des épaisseurs variables*. Thierry Pudet. January 1993.

Research Report 24: *Programmable Active Memories: a Performance Assessment*. Patrice Bertin, Didier Roncin, and Jean Vuillemin. March 1993.

Research Report 25: *On Circuits and Numbers*. Jean Vuillemin. April 1993.

Research Report 26: *Numerical Valuation of High Dimensional Multivariate European Securities*. Jérôme Barraquand. March 1993.

Research Report 27: *A Database Interface for Complex Objects*. Marcel Holsheimer, Rolf A. de By, and Hassan Aït-Kaci. March 1993.

Research Report 28: *Feature Automata and Recognizable Sets of Feature Trees*. Joachim Niehren and Andreas Podelski. March 1993.

Research Report 29: *Real Time Fitting of Pressure Brushstrokes*. Thierry Pudet. March 1993.

Research Report 30: *Rollit: An Application Builder*. Solange Karsenty and Chris Weikart. April 1993.

# 26

## Numerical Valuation of High Dimensional Multivariate European Securities

Jérôme Barraquand

**digital**

**PARIS RESEARCH LABORATORY**

85, Avenue Victor Hugo  
92563 RUEIL MALMAISON CEDEX  
FRANCE